International Journal of Theoretical Physics, Vol. 28, No. 8, 1989

LRS Bianchi Type I Perfect Fluid Solutions Generated from Known Solutions

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Received September 20, 1988

Some LRS Bianchi type I perfect fluid solutions are generated from known solutions of this type. The solutions represent spatially homogeneous and anisotropic cosmological models which would give essentially empty space for large time. The physical and kinematic properties of the models are discussed.

1. INTRODUCTION

Hajj-Boutros (1985) presented a generating technique which converts known LRS Bianchi type I perfect fluid solutions into new solutions of the same type. For the metric

$$ds^2 = -dt^2 + A^2\,dx^2 + B^2(dy^2 + dz^2) \tag{1}$$

where $A(t)$ and $B(t)$ are cosmic scale functions, he derived formulas for generating new functions A and B:

$$A = A_0 \exp\left\{\int \left[(BA_0^2)\left(\int \frac{dt}{BA_0^2} + k_1\right)\right]^{-1} dt + k_2\right\} \tag{2}$$

and

$$B = B_0 \exp\left\{\int \left[\frac{B_0^4}{A}\left(\int \frac{2A}{B_0^4}\,dt + k_3\right)\right]^{-1} dt + k_4\right\} \tag{3}$$

The k's are integration constants. Formula (2) converts the known functions $[A_0, B]$ into new functions $[A, B]$ where B stays invariable. Formula (3) converts the known functions $[A, B_0]$ into new functions where A remains invariable. Starting from the general Kasner stiff-matter and vacuum solutions (Jacobs, 1968), I recently obtained new classes of LRS Bianchi type I models filled with perfect fluids (Shri Ram, 1988).

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0020-7748/89/0800-0917$06.00/0

Starting from the solutions of Hajj-Boutros and Sfeila (1987) and using the above technique, I here obtain new classes of perfect fluid solutions. The solutions represent anisotropic cosmological models which would give essentially empty universes for large time. The physical and kinematic properties of solutions are studied in detail.

2. GENERATED SOLUTIONS

For the metric (1), the Einstein field equations

$$R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu}=KT_{\mu\nu},\qquad K=8\pi \tag{4}$$

are written as

$$\frac{2B''}{B}+\left(\frac{B'}{B}\right)^2=-Kp \tag{5}$$

$$\frac{B''}{B}+\frac{A''}{A}+\frac{A'B'}{AB}=-Kp \tag{6}$$

$$2\frac{A'B'}{AB}+\left(\frac{B'}{B}\right)^2=K\rho \tag{7}$$

where ρ is the energy density and p is the isotropic pressure. A prime denotes derivative with respect to t.

2.1. Solutions Generated From Dust-Filled Model

The LRS Bianchi type I dust-filled solution of Hajj-Boutros and Sfeila (1987) is given by the metric

$$ds^2=-dt^2+t^{4/3}(c_2-c_1t^{-1})^2\,dx^2+t^{4/3}(dy^2+dz^2) \tag{8}$$

c_1, c_2 are arbitrary constants. Applying the formula (3), we obtain

$$B=mt^{-1/3}(lt^2-2c_2t+c_1)^{1/2} \tag{9}$$

where l and m are integration constants. By the change of scale, the metric of the generated solution is

$$ds^2=-dt^2+t^{4/3}(c_2-c_1t^{-1})^2\,dx^2+t^{-2/3}(lt^2-2c_2t+c_1)(dy^2+dz^2) \tag{10}$$

Inserting now the new values of A and B into (5) and (7), we obtain

$$Kp=\frac{(2lc_1-c_2^2)t^2-2c_1c_2t+c_1^2}{t^2(lt^2-2c_2t+c_1)^2} \tag{11}$$

$$K\rho=\frac{1}{9}\left[\frac{2(2c_2t+c_1)(2lt^2-c_2t-c_1)}{t^2(c_2t-c_1)(lt^2-2c_2t+c_1)}+\frac{(2lt^2-c_2t-c_1)^2}{t^2(lt^2-2c_2t+c_1)^2}\right] \tag{12}$$

For kinematic parameters see Ellis (1971) and Collins and Wainwright (1983). The nonvanishing components of the shear tensor $\sigma_{\mu\nu}$ are given by

$$\sigma_{11}=\frac{2(lc_1-c_2)^2(c_2t-c_1)t^2}{3t^{5/3}(lt^2-2c_2t+c_1)} \tag{13}$$

$$\sigma_{22}=\sigma_{33}=\frac{(c_2^2-lc_1)t^2}{3t^{5/3}(lt^2-c_2t+c_1)} \tag{14}$$

From (13) and (14) it is clear that the shear tensor is nonzero for all values of t $(0<t<\infty)$; thus, the model (10) is anisotropic. For the model all of the fluids are acceleration and rotation-free, but they do have expansion θ and shear scalar σ given by

$$\theta=\frac{4lc_2t^2-(lc_1+5c_2^2)t+2c_1^2}{3t(c_2t-c_1)(lt^2-2c_2t+c_1)} \tag{15}$$

$$\sigma=\frac{(lc_1-c_2^2)t[(2l+c_2^2)t^2-(4c_2+2c_1c_2)t+2c_1+c_1^2]^{1/2}}{3(c_2t-c_1)(lt^2-2c_2t+c_1)^{3/2}} \tag{16}$$

At $t=0$, the proper volume tends to zero and ρ, p, θ, and σ approach infinitely large magnitudes. The model, exploding from the singularity state $t=0$, approaches to infinite volume as $t\to\infty$. In this limit, ρ, p, θ, and σ all vanish; thus, the model gives essentially empty space for large t. The ratio σ/θ tends to zero as $t\to\infty$ and therefore the universe ultimately becomes isotropic.

Applying the formula (2) to the metric (8), we get the metric of solution of the form

$$ds^2=-dt^2+t^{-2/3}(t+k)^2\,dx^2+t^{4/3}(dy^2+dz^2) \tag{17}$$

k is an arbitrary constant. For the model (17), the pressure, density, and nonzero kinematic parameters are

$$p=0 \tag{18}$$

$$k\rho=\frac{4}{9t^2}\left(\frac{2t-k}{t+k}\right) \tag{19}$$

$$\sigma_{11}=-\frac{2k(t+k)}{3t^{5/3}} \tag{20}$$

$$\sigma_{22}=\sigma_{33}=\frac{kt^{1/3}}{3(t+k)} \tag{21}$$

$$\theta=\frac{2t+k}{t(t+k)} \tag{22}$$

$$\lim_{t\to\infty}\frac{\sigma}{\theta}=0 \tag{23}$$

The physical and kinematic behavior of this dust-filled model are same as that of the model (10).

2.2. Solutions Generated from Perfect Fluid Model

Let us consider the perfect fluid solution of Hajj-Boutros and Sfeila (1987) with metric

$$ds^2 = -dt^2 + t(c_2 - 2c_1 t^{-1/2})^2\, dx^2 + t(dy^2 + dz^2) \tag{24}$$

Inserting

$$A = t^{1/2}(-2c_1 t^{-1/2} + c_2), \qquad B_0 = t^{1/2} \tag{25}$$

into (3) and integrating, we obtain

$$B = c_4(c_1 - c_2 t^{1/2} + c_3 t)^{1/2} \tag{26}$$

The c's are arbitrary constants. The metric of the generated solution can be written in the form

$$ds^2 = -dt^2 + t(-2c_1 t^{-1/2} + c_2)^2\, dx^2 + (c_1 - c_2 t^{1/2} + c_3 t)\,(dy^2 + dz^2) \tag{27}$$

The pressure, density, and nonzero kinematic parameters are obtained as

$$Kp = \frac{8c_2c_3t^{1/2} + 12c_3^2 t + 4c_1c_2t^{-1/2} - 5c_2^2}{16(c_3t^{3/2} - c_2t + c_1t^{1/2})^2} \tag{28}$$

$$k\rho = \frac{4c_2c_3^2t^2 + 8c_3^2t^2 + 4(c_2^2c_3 - c_3^2c_1 - 2c_2c_3)t^{3/2}}{(c_2t - 2c_1t^{1/2})(c_3t^{3/2} - c_2t + c_1t^{1/2})^2} \tag{29}$$

$$\theta = \frac{2c_2c_3t^{3/2} - 2(c_2^2 + 2c_1c_3)t + 3c_1c_2t^{1/2}}{2(c_2t - 2c_1t^{1/2})(c_3t^{3/2} - c_2t + c_1t^{1/2})} \tag{30}$$

$$\sigma_{11} = \frac{(4c_1c_3 - c_2^2)(c_2^2t^2 + 4c_1t - 4c_1c_2t^{3/2})}{2(c_2t - 2c_1t^{1/2})(c_3t^{3/2} - c_2t + c_1t^{1/2})} \tag{31}$$

$$\sigma_{22} = \sigma_{33} = \frac{(c_2^2 - 4c_1c_3)(c_1t - c_2t^{3/2} + c_3t^2)}{12(c_2t - 2c_1t^{1/2})(c_3t^{3/2} - c_2t + c_1t^{1/2})} \tag{32}$$

The lengthy expression for σ is not given. It is seen that the ratio $\sigma/\theta \to 0$ as $t \to \infty$. The physical and kinematic properties are same as that of the model (10).

Applying the formula (2), we arrive at the metric (24) with different parameters.

3. CONCLUSIONS

I have presented some new LRS Bianchi type I models for which all of the fluids are acceleration and rotation free, but they do have expansion and shear. For large time, the models would give essentially empty spaces. As far as I know, these solutions are new and can be added to the rare perfect fluid solutions of this type existing in the literature not satisfying the equation of state.

ACKNOWLEDGMENT

I am extremely grateful to Dr. K. P. Singh for valuable suggestions and continued encouragement.

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